Solution Sheet 7

Exercise 1. Let $b, c \in \mathbb{C}$ with $\Delta = b^2 - 4c \neq 0$. Consider the projective curve C defined by

$$Y^2 = X^2 + bXZ + cZ^2.$$

(1) Show that C is smooth.
(2) On the affine chart $Z = 1$ (so $x = X/Z,\ y = Y/Z$) define $t := y - x$. Solve for x and y in terms of t, and deduce a rational parametrization $t \mapsto (x(t), y(t))$ of C. Show that this extends to a biholomorphism $\Phi : \mathbb{P}^1 \to C$. **Hint:** Homogenize $t = u/v$ and verify that

$$\Phi([u:v]) = \left[cv^2 - u^2 : u^2 - buv + cv^2 : 2uv - bv^2\right]$$

has inverse $\Psi([X:Y:Z]) = [Y - X : Z]$.
(3) Use the rational parametrization from (2) to deduce Euler's formula

$$\int \frac{dx}{\sqrt{x^2 + bx + c}} = -\log|2t - b| + C,$$

with $t = \sqrt{x^2 + bx + c} - x$.

Solution 1.

(1) Let $F = Y^2 - (X^2 + bXZ + cZ^2)$. Then

$$F_X = -2X - bZ, \qquad F_Y = 2Y, \qquad F_Z = -bX - 2cZ.$$

If $Z \neq 0$, then $F_Y = 0$, thus $Y = 0$. Plugging into $F = 0$ gives $X^2 + bXZ + cZ^2 = 0$ with simple roots when $\Delta = b^2 - 4c \neq 0$, thus $(F_X, F_Y, F_Z) \neq (0, 0, 0)$. If $Z = 0$, then $F = Y^2 - X^2 = 0$ implies that $[X : Y : 0] = [1 : \pm 1 : 0]$. At those points $F_Y = \pm 2 \neq 0$ or $F_X = -2 \neq 0$. Hence C is smooth.

(2) On the affine chart $Z = 1$ define $t := y - x$. From

$$(y - x)(y + x) = bx + c,$$

we get $(2x + t)t = bx + c$. From this we obtain

$$x = \frac{c - t^2}{2t - b}, \qquad y = x + t = \frac{t^2 - bt + c}{2t - b}. \tag{1}$$

So the rational parametrization is

$$t \mapsto (x(t), y(t)) = \left(\frac{c - t^2}{2t - b}, \frac{t^2 - bt + c}{2t - b}\right).$$

To homogenize this, set $t = u/v$, and this then defines a map $\Phi : \mathbb{P}^1 \to C$ by

$$\Phi([u:v]) = \left[cv^2 - u^2 : u^2 - buv + cv^2 : 2uv - bv^2\right].$$

Define

$$\Psi : C \to \mathbb{P}^1, \qquad [X:Y:Z] \mapsto [Y - X : Z].$$

Then we can check that $\Psi \circ \Phi([u:v]) = [u:v]$. On $Z = 1$, we can also verify that $\Phi \circ \Psi([x:y:1]) = [x:y:1]$. Thus Φ and Ψ are mutual inverses on the dense open subset $Z = 1$. By the identity theorem for holomorphic maps between compact Riemann surfaces they agree globally. Therefore $\mathbb{P}^1$ and C are biholomorphic.

(3) From the two parametrizations in (1) we deduce that

$$\frac{dx}{dt} = -\frac{2y}{2t-b}.$$

Since $y = \sqrt{x^2+bx+c}$ on C and $t = y - x = \sqrt{x^2+bx+c} - x$ we obtain

$$\int \frac{dx}{\sqrt{x^2+bx+c}} = \int \frac{dx}{y} = -\int \frac{2dt}{2t-b} = -\log|2t-b| + C.$$

Exercise 2. Let $F(X,Y,Z) = ZY^2 - X^3$ and consider the associated projective curve $C \subset \mathbb{P}^2$. Let $\nu : \tilde{C} \to C$ be the normalization of C.

(1) What are the singular points of C?
(2) Show that the normalization $\tilde{C}$ is biholomorphic to $\mathbb{P}^1$.

Solution 2.

(1) Computing partial derivatives gives

$$F_X = -3X^2, \quad F_Y = 2ZY, \quad F_Z = Y^2.$$

The only point where all three vanish is $[0:0:1]$.

(2) Define

$$\Phi : \mathbb{P}^1 \to C, \quad [u:v] \mapsto [u^2v : u^3 : v^3].$$

This is holomorphic and lands in C since $(u^3)^2v^3 = (u^2v)^3$. Now define a map on the smooth curve $C_{\mathrm{sm}} = C \backslash \{[0:0:1]\}$ by

$$\Psi : C_{\mathrm{sm}} \to \mathbb{P}, \quad [X:Y:Z] \mapsto [Y:X].$$

Then Φ restricts to a biholomorphism between $\mathbb{P}^1 \backslash \{[0:1]\}$ and C_{sm}. Indeed:

- For any $[u:v] \in \mathbb{P}^1 \backslash \{[0:1]\}$ we have

$$\Psi \circ \Phi([u:v]) = \Psi([u^2v : u^3 : v^3]) = [u^3 : u^2v] = [u:v].$$

- For any $[X:Y:Z] \in C_{\mathrm{sm}}$ with $Z = 1$ we have

$$\Phi \circ \Psi([X:Y:1]) = \Phi([y:x]) = [y^2x : y^3 : x^3] = [x : y : x^3/y^3] = [X:Y:1].$$

Also we have $\Phi \circ \Psi([0:1:0]) = \Phi([1:0]) = [0:1:0]$.

The normalization $\nu : \tilde{C} \to C$ is the unique holomorphic map from a compact Riemann surface which is a biholomorphism over C_{sm}. The map Φ already has this property. It then follows that there is a unique biholomorphism $\psi : \mathbb{P}^1 \to \tilde{C}$ with $\nu \circ \psi = \Phi$.

Exercise 3. (for credit, due on 16 November)

Let $n \geq 4$ be even and $a_1, \ldots, a_n \in \mathbb{C}$ be nonzero and distinct. Consider the projective curve $C \subset \mathbb{P}^2$ defined by

$$F(X,Y,Z) = Y^2Z^{n-2} - \prod_{i=1}^{n}(X - a_iZ) = 0.$$

(1) *(1 point)* Show that $P_\infty = [0:1:0]$ is the only singular point of C.

Let $\nu : \tilde{C} \to C$ be the normalization of C. Consider the holomorphic map $\pi : \tilde{C} \to \mathbb{P}^1$ defined by

$$Q \mapsto \begin{cases} [X:Z] & \text{if } Q = [X:Y:Z] \in C \setminus \{P_\infty\} \cong \tilde{C} \setminus \nu^{-1}(P_\infty); \\ [1:0] & \text{if } Q \in \nu^{-1}(P_\infty). \end{cases}$$

(2) *(1 point)* Compute the ramification points of π on $C \setminus \{P_\infty\} \cong \tilde{C} \setminus \nu^{-1}(P_\infty)$.

On $\tilde{C}$ we define the meromorphic functions

$$u = \left(\frac{Z}{X}\right) \circ \nu, \quad v = \left(\frac{YZ^{n/2-1}}{X^{n/2}}\right) \circ \nu.$$

(3) *(1 point)* Show that for every $Q \in \nu^{-1}(P_\infty)$ we have $u(Q) = 0$. **Hint:** Work in the chart $Y \neq 0$ and set $\xi = X/Y$ and $\zeta = Z/Y$. Deduce from the curve equation $1/\zeta^2 = \prod(1/u - a_i)$ that $u \to 0$ as we approach $(\xi, \zeta) = (0, 0)$.

(4) *(1 points)* Show that we can write the curve equation as

$$v^2 = \prod_{i=1}^{n}(1 - a_i u) =: G(u),$$

and deduce that at any $Q \in \tilde{C}$ with $u(Q) = 0$ we must have $v(Q) \in \{+1, -1\}$.

We now argue why the fiber $\pi^{-1}([1:0])$ actually consists of these two points. Let D be the affine curve D defined by

$$\Phi(u, v) = v^2 - G(u) = 0.$$

At $(u, v) = (0, \pm 1)$ we have $\Phi(0, \pm 1) = 0$ and $\partial_v \Phi(0, \pm 1) = 2(\pm 1) \neq 0$. By the implicit function theorem there exist a neighborhood U around 0 and holomorphic branches $v = \phi_\pm(u)$ on U with $\phi_\pm(0) = \pm 1$ and $\Phi(u, \phi_\pm(u)) = 0$. Thus, near $u = 0$, D is the disjoint union of the two graphs $v = \phi_+(u)$ and $v = \phi_-(u)$. Intersecting with the line $u = 0$ gives exactly the two points $(0, 1)$ and $(0, -1)$ on D, which are smooth. Normalization is a biholomorphism over the smooth locus, hence there are $Q_\pm \in \widetilde{C}$ mapping to $(0, \pm 1)$. Therefore the fiber $\pi^{-1}([1:0])$ consists of two points $Q_\pm$, both of which are unramified (since the degree is 2).

(5) *(1 point)* Apply Riemann-Hurwitz to π to determine $g(\tilde{C})$.

Solution 3.

(1) In the affine chart $Z = 1$, the curve equation is $f(x, y) = y^2 - \prod_{i=1}^n (x - a_i) = 0$. From $f_y = 2y$, any singular affine point must satisfy $y = 0$. Then $f(x, 0) = 0$ forces $x = a_k$ for some k. But

$$f_x(a_k, 0) = -\prod_{j \neq k}(a_k - a_j) \neq 0$$

since the a_i are distinct. Thus there are no affine singularities. At infinity, $Z = 0$ forces $X = 0$, so the only candidate is $P_\infty = [0:1:0]$. One can check that

$$F_X(P_\infty) = 0, \quad F_Y(P_\infty) = 0.$$

We have

$$F_Z = (n-2)Y^2 Z^{n-3} + \sum_{k=1}^{n} a_k \prod_{j \neq k}(X - a_j Z).$$

If $n = 3$, then $F_Z(P_\infty) \neq 0$, so P_∞ is a smooth point. If $n \geq 4$, then $F_Z(P_\infty) = 0$, so P_∞ is a singular point.

(2) Let's analyse the fiber over a finite point $[\lambda : 1]$. A point $[X : Y : Z]$ maps to $[\lambda : 1]$ if and only if there exists $s \in \mathbb{C}^\times$ with $(X, Z) = s(\lambda, 1)$ if and only if $X = \lambda Z$ and $Z \neq 0$. Thus the entire fiber over $[\lambda : 1]$ lies in the affine chart $Z = 1$. In coordinates $x = X/Z, y = Y/Z^2$, the curve is $y^2 = \prod_{i=1}^n (x - a_i)$. The condition $X = \lambda Z$ becomes $x = \lambda$, so points in the fiber satisfy $y^2 = \prod_{i=1}^n (\lambda - a_i)$. We can now read off the fiber:

- If $\lambda \notin \{a_1, \dots, a_n\}$, then
$$\pi^{-1}([\lambda : 1]) = \left\{[\lambda : \pm\sqrt{\prod(\lambda - a_i)} : 1]\right\}.$$
- If $\lambda = a_k$ for some k, then
$$\pi^{-1}([a_k : 1]) = [a_k : 0 : 1]$$
is a ramification point of multiplicity 2.

Since all those affine points are smooth, the fiber on $\tilde{C}$ equals the one we just computed on C.

(3) We work in the affine chart $Y = 1$ and set $\xi = X/Y$ and $\zeta = Z/Y$. Then $P_\infty = [0 : 1 : 0]$ corresponds to $(\xi, \zeta) = (0, 0)$, and the curve equation is
$$\zeta^{n-2} = \prod_{i=1}^{n} (\xi - a_i\zeta).$$
Note that $\zeta = 0$ implies that $\xi^n = 0$, that is, $(\xi, \zeta) = (0, 0)$. Thus on each component of C near $(0, 0)$ we have $\zeta \neq 0$, so dividing by ζ^n gives
$$\frac{1}{\zeta^2} = \prod_{i=1}^{n} \left(\frac{\xi}{\zeta} - a_i\right).$$
With $u = (Z/X) \circ \nu = \zeta/\xi$ we can write this as
$$\frac{1}{\zeta^2} = \prod_{i=1}^{n} \left(\frac{1}{u} - a_i\right).$$
If $(\xi, \zeta) \to (0, 0)$ along the curve, we have $\left|1/\zeta^2\right| \to \infty$. The product on the right hand side can blow up only if $u \to 0$. Hence $u(Q) = 0$ for every $Q \in \nu^{-1}(P_\infty)$.

(4) From
$$v^2 = \prod_{i=1}^{n} (1 - a_i u).$$
we see that $u = 0$ implies $v = \pm 1$.

(5) The degree of π is 2. Applying the Riemann-Hurwitz formula to the map $\pi : \tilde{C} \to \mathbb{P}^1$ yields
$$g(\tilde{C}) = \frac{n-2}{2}.$$

Exercise 4. Determine the fundamental group of a closed surface of genus g with $n \geq 1$ distinct punctures.

Solution 4.
Let us realize Σ_g by a $4g$–gon P with the usual edge identifications. The image of ∂P in Σ_g is a wedge of the $2g$ loops $a_1, b_1, \dots, a_g, b_g$ based at the vertex v (all polygon vertices identify to v). Let $S = \Sigma_g \setminus \{p_1, \dots, p_n\}$. We now remove disjoint open disks D_i around the punctures. The resulting surface $\Sigma_g^n = \Sigma_g \setminus \bigcup_i \mathrm{int}(D_i)$ is a deformation retract of S. Now we choose $n - 1$ disjoint arcs $\alpha_2, \dots, \alpha_n$ from v to the boundaries $\partial D_2, \dots, \partial D_n$. Then for each $i = 2, \dots, n$ we define a loop β_i based at v:
$$\beta_i = \alpha_i \cdot (\text{loop once around } \partial D_i) \cdot \alpha_i^{-1}.$$
Then we set
$$K = \big(a_1 \vee b_1 \vee \cdots \vee a_g \vee b_g\big) \vee \beta_2 \vee \cdots \vee \beta_n \subset \Sigma_g^n.$$

This K is a bouquet of $2g + n - 1$ circles. Now observe that Σ_g^n deformation retracts onto K. Therefore $\pi_1(S) \cong \pi_1(\Sigma_g^n) \cong \pi_1(K)$. Thus the fundamental group of a closed surface of genus g with $n \geq 1$ punctures is the free group of rank $2g + n - 1$.